Calculating adiabatic evolution of the perturbed DNLS/MNLS solitons

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Abstract

A symbolic computation technique is developed to calculate adiabatic evolution equations for parameters of the perturbed DNLS/MNLS solitons obtained by the recently developed direct perturbation theory [X.-J. Chen and J. Yang, Phys. Rev. E **65**, 066608(2002)]. Effects of the intrapulse Raman scattering, third-order group velocity dispersion, and narrow-banded filters with nonlinear gain on the MNLS solitons are studied as examples.

Key words: DNLS/MNLS solitons, perturbation, symbolic computation
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1 Introduction

It has been well known that exactly integrable nonlinear differential equations support soliton solutions which travel stationarily and collide elastically. It was also known that the physical situations giving rise to exactly solvable equations are highly idealized. Perturbations violating their integrabilities actually exist. If these perturbations are small enough, their influence on solitons can still be known analytically by perturbation theories for solitons(see, e.g., [1,2,3]). In the picture depicted by perturbation theories, the lowest approximation is an adiabatic solution in which the soliton keep its profile unchanged while its parameters such as amplitude, velocity, initial center and initial phase may evolve adiabatically[1,2]. Perturbations can also induce non-adiabatic changes

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such as radiation emissions. A mathematically complete perturbation theory can obtain not only evolution equations for parameters of solitons analytically but also a formula for calculating the perturbation-induced radiation emission.

With vanishing boundary conditions, $u \to 0$ as $|x| \to \infty$, the derivative nonlinear Schrödinger(DNLS) equation,

$$iu_t + u_{xx} + i(|u|^2 u)_x = 0, \tag{1}$$

is an integrable model describing small amplitude nonlinear Alfvén waves in a low-β (the ratio of kinetic to magnetic pressure) plasma, propagating strictly parallel to the ambient magnetic field. Here u, x, and t represent the transverse complex magnetic field, the normalized longitudinal coordinate, and the normalized time, respectively and the subscripts denote partial derivatives[4,5,6].

The modified nonlinear Schrödinger (MNLS) equation,

$$iq_z + \frac{1}{2}q_{tt} + is(|q|^2 q)_t + |q|^2 q = 0, \tag{2}$$

is an integrable model in describing propagation of femtosecond pulses in single mode fibers [7,8,9,10]. For picosecond pulses, it is well known that the nonlinear Schrödinger (NLS, the case when $s = 0$) equation is a good model. The MNLS model includes the nonlinear dispersion term(the third term on the left) which is one of the several higher order effects becoming more significant in the femtosecond region[11,12]. Here u, z, t, and s denote the normalized electric field, the normalized distance, the normalized time measured in the frame of reference moving with the pulse, and the relative amplitude of the nonlinear dispersion, respectively. We assume $s \geq 0$ in this letter because the case for $s < 0$ can be obtained by a simple transformation $t \to -t$. The DNLS equation and the MNLS equation are connected by a gauge-like transformation(see, e.g. [13]). The soliton solution for the DNLS with vanishing boundary conditions was found by the inverse scattering transform technique[14] and the soliton solution for the MNLS can be found by the gauge-like transformation.

Perturbation theory for the DNLS/MNLS solitons was first developed by a method based on the Riemann-Hilbert problem[15], which had minor errors in evolution equations for the initial center and phase and gave no correction beyond the adiabatic approximation. Later a direct perturbation method[16] corrected these errors and obtained a formula for calculating the perturbation-induced radiation emission. With well established perturbation theories, influence of higher order effects and methods of controlling soliton shapes and frequency shift against perturbations can be studied. However, evolution equations for the perturbed MNLS solitons are much more complicated than those for the perturbed NLS solitons[1,2]. Problems solved by perturbation theories

for DNLS/MNLS solitons[15,16] were limited to some relatively simple cases so far[15,16,17]. Symbolic technique seems helpful for further studies. In this letter, we develop a symbolic technique, which is effective for usual perturbations, to calculate adiabatic evolution equations for parameters of perturbed DNLS/MNLS solitons automatically. In the MNLS model, only one higher order effect beyond the NLS model, the nonlinear dispersion, is considered. Other higher order effects, such as the intrapulse Raman scattering and the third-order dispersion, should be treated as perturbations. Evolution equations for parameters of the MNLS soliton under some control method, such as narrow-banded filter and nonlinear gain, will be valuable in estimating these methods. In section 2, we rewrite those evolution equations obtained in Ref.[16] in slightly simpler forms. In section 3, we describe the symbolic technique. In section 4, we calculate evolution equations of the MNLS solitons under intrapulse Raman scattering with local approximation, the third-order group velocity(GVD) dispersion, and narrow band filters with nonlinear gain. In small s limit, all these results approach those for the NLS solitons in the literature, as generally shown in Ref.[16].

2 Adiabatic evolution of parameters of the perturbed DNLS/MNLS solitons

2.1 Perturbed DNLS solitons

The DNLS one-soliton[14] can be rewritten as

$$u_s(x,t) = -2\Delta\sin(2\beta)\frac{\cosh(\theta - i\beta)}{\cosh^2(\theta + i\beta)}\exp(-i\varphi), \tag{3}$$

where

$$\theta(x,t) = 2\eta(x + 4\xi t - x_0), \tag{4}$$

$$\varphi(x,t) = 2\frac{\xi}{\eta}\theta - 4(\xi^2 + \eta^2)t + \varphi_0, \tag{5}$$

$$\xi = \Delta^2\cos(2\beta), \quad \eta = \Delta^2\sin(2\beta), \quad 0 < \beta < \pi/2. \tag{6}$$

Its amplitude is

$$A = 4\Delta\sin\beta. \tag{7}$$

Simple calculations yield its energy,

$$E = \int_{-\infty}^{+\infty} dx |u_s|^2 = 8\beta, \tag{8}$$

and the FWHM width,

$$w = \frac{1}{\Delta^2 \sin(2\beta)} \ln\left(\cos\beta + \sqrt{1+\cos^2\beta}\right). \tag{9}$$

In presence of perturbations, the zero on the left hand side of Eq. (1) should be replaced with the perturbation function $ir(u)$. As β is proportional to the energy, considering evolution of β and Δ, instead of ξ and η in Ref. [16], is more convenient. Evolution equations for parameters of the DNLS solitons in Ref. ([16]) are reformulated as,

$$\frac{d\beta}{dt} = \frac{1}{16\eta} \int_{-\infty}^{\infty} d\theta R_+(\theta, t), \tag{10}$$

$$\frac{d\Delta}{dt} = i\frac{\Delta}{16\eta} \int_{-\infty}^{+\infty} d\theta \tanh(\theta + i\beta) R_+(\theta, t), \tag{11}$$

$$\frac{dx_0}{dt} = \frac{\Delta^2}{16\eta^3} \int_{-\infty}^{+\infty} d\theta\theta \frac{\cosh(\theta + i3\beta)}{\cosh(\theta + i\beta)} R_-(\theta, t) - \frac{i}{32\eta^2} \int_{-\infty}^{\infty} d\theta R_-(\theta, t), \tag{12}$$

$$\frac{d\varphi_0}{dt} = \frac{\Delta^4}{8\eta^3} \int_{-\infty}^{+\infty} d\theta\theta R_-(\theta, t) + i\frac{\Delta^2}{16\eta^2} \int_{-\infty}^{+\infty} d\theta \frac{\cosh(\theta - i\beta)}{\cosh(\theta + i\beta)} R_-(\theta, t), \tag{13}$$

where

$$R_\pm(\theta, t) = \overline{u_0(\theta)}[r_0(\theta, t) \pm \overline{r_0(-\theta, t)}], \tag{14}$$

$$u_0(\theta) = u_s \exp(i\varphi), \quad r_0(\theta, t) = r \exp(i\varphi), \tag{15}$$

and the bar stands for complex conjugate in this letter.

2.2 Perturbed MNLS solitons

The MNLS one-soliton is

$$q_s(t,z) = -2\Delta\sin(2\beta)\frac{\cosh(\theta - i\beta)}{\cosh^2(\theta + i\beta)}\exp(-i\varphi), \tag{16}$$

where

$$\theta(t,z) = 4s\eta(t - t_0 - z/v), \tag{17}$$

$$\varphi(t,z) = 2\frac{\xi}{\eta}\theta - 8s^2(\xi^2 + \eta^2)z - \frac{t}{s} + \frac{z}{2s^2} + \varphi_0, \tag{18}$$

$$\xi = \Delta^2\cos(2\beta), \quad \eta = \Delta^2\sin(2\beta), \quad 0 < \beta < \pi/2, \tag{19}$$

and $v = s/(1 - 4s^2\xi)$ is its velocity. Similar to the DNLS soliton, we have amplitude,

$$A = 4\Delta\sin\beta, \tag{20}$$

energy,

$$E = \int_{-\infty}^{+\infty} dt|q_s|^2 = \frac{4\beta}{s}, \tag{21}$$

and the FWHM width,

$$\tau = \frac{1}{2s\Delta^2\sin(2\beta)}\ln\left(\cos\beta + \sqrt{1+\cos^2\beta}\right), \tag{22}$$

of the MNLS soliton. Also, in presence of perturbations, the zero on the right hand side of Eq. (2) should be replaced with the perturbation function $ir(q)$. Evolution equations for MNLS soliton parameters in Ref. [16] are reformulated as,

$$\frac{d\beta}{dz} = \frac{1}{16\eta}\int_{-\infty}^{+\infty} d\theta R_+(\theta,z), \tag{23}$$

$$\frac{d\Delta}{dz} = i\frac{\Delta}{16\eta}\int_{-\infty}^{+\infty} d\theta\tanh(\theta + i\beta)R_+(\theta,z), \tag{24}$$

$$\frac{dt_0}{dz} = \frac{\Delta^2}{32s\eta^3}\int\limits_{-\infty}^{+\infty} d\theta\theta\frac{\cosh(\theta+i3\beta)}{\cosh(\theta+i\beta)}R_-(\theta,z) - \frac{i}{64s\eta^2}\int\limits_{-\infty}^{+\infty} d\theta R_-(\theta,z), \quad (25)$$

$$\frac{d\varphi_0}{dz} = \frac{\Delta^4}{8\eta^3}\int\limits_{-\infty}^{+\infty} d\theta\theta R_-(\theta,z) + i\frac{\Delta^2}{16\eta^2}\int\limits_{-\infty}^{+\infty} d\theta\frac{\cosh(\theta-i\beta)}{\cosh(\theta+i\beta)}R_-(\theta,z), \quad (26)$$

where

$$R_\pm(\theta,z) = \overline{q_0(\theta)}[r_0(\theta,z) \pm \overline{r_0(-\theta,z)}], \quad (27)$$

$$q_0(\theta) = q_s\exp(i\varphi), \quad r_0(\theta,z) = r\exp(i\varphi). \quad (28)$$

3 A symbolic technique to calculate integrals in evolution equations for soliton parameters

In general, integrals in evolution equations for perturbed DNLS/MNLS solitons are rather complicated. Even modern symbolic softwares can not always tackle them directly. However, taking the perturbed MNLS solitons as an example, most perturbation functions can be expressed as

$$r(q) = \sum_{k=0}^{n} r_k(|q|^2)\partial_t^k q, \quad (29)$$

where $r_k(|q|^2)$ are complex functions[11,12]. In what follows we will show that for this category of perturbation functions integrals in the evolution equations can be systematically solved by the technique of residue theorem.

For perturbation functions in category of Eq. (29), there are only four types of integrands in the evolution equations needed to be tackled. Having been continuated to the whole complex plane of θ, they are

(1) $f(\theta)$ with $f(\theta+i\pi) = -f(\theta)$,
(2) $g(\theta)$ with $g(\theta+i\pi) = g(\theta)$,
(3) $F(\theta) = \theta f(\theta)$,
(4) $G(\theta) = \theta g(\theta)$.

Within the closed path shown in Fig.1, all of them have and only have a pair of singularities $\theta_\pm = i(\pi/2 \pm \beta)$. Integrations of them on the two vertical line segments are obviously zero. Using the residue theorem on the closed path

shown in Fig. 1, we get

$$\int_{-\infty}^{\infty} f(\theta)d\theta = i\pi\{\mathrm{Res}[f(\theta_+)] + \mathrm{Res}[f(\theta_-)]\}, \tag{30}$$

and

$$\int_{-\infty}^{\infty} F(\theta)d\theta = -i\frac{\pi}{2}\int_{-\infty}^{\infty} f(\theta)d\theta + i\pi\{\mathrm{Res}[F(\theta_+)] + \mathrm{Res}[F(\theta_-)]\}. \tag{31}$$

For $g(\theta)$ and $G(\theta)$, by introducing two auxiliary functions,

$$h(\theta,\rho) = \exp(i\rho\theta)g(\theta), \tag{32}$$

and

$$H(\theta,\rho) = \exp(i\rho\theta)G(\theta), \tag{33}$$

in which $\rho > 0$, we also have

$$\int_{-\infty}^{\infty} g(\theta)d\theta = i2\pi \lim_{\rho\to 0^+} \frac{\mathrm{Res}[h(\theta_+,\rho)] + \mathrm{Res}[h(\theta_-,\rho)]}{1-\exp(-\rho\pi)}, \tag{34}$$

$$\begin{aligned}\int_{-\infty}^{\infty} G(\theta)d\theta = \lim_{\rho\to 0^+} \Big\{ &-\pi^2 \frac{\mathrm{Res}[h(\theta_+,\rho)] + \mathrm{Res}[h(\theta_-,\rho)]}{2\sinh^2(\rho\pi/2)} \\ &+ i2\pi \frac{\mathrm{Res}[H(\theta_+,\rho)] + \mathrm{Res}[H(\theta_-,\rho)]}{1-\exp(-\rho\pi)} \Big\}. \end{aligned} \tag{35}$$

Therefore, despite the fact that these integrals may not be tackled directly, calculation of them comes down to calculation of residues and limitations which can always be done symbolically by modern commercial mathematical softwares.

4 Examples of perturbed MNLS solitons

4.1 Intrapulse Raman scattering

In local approximation, the intrapulse Raman scattering was described by a perturbation function[11,12],

$$r(q) = -\sigma_R q(|q|^2)_t, \tag{36}$$

where σ_R is a constant. We get $R_+ = 2\bar{q}_0 r_0$, $R_- = 0$. This problem is so simple that we don't need the symbolic technique in the preceding section. Direct integrations yields

$$\frac{dt_0}{dz} = 0, \quad \frac{d\varphi_0}{dz} = 0, \tag{37}$$

$$\frac{d\beta}{dz} = 0, \tag{38}$$

$$\frac{d\Delta}{dz} = -\frac{32}{3} s\sigma_R \Delta^5 (2 + \cos^2 2\beta - 6\beta \cot 2\beta) \tag{39}$$

It is obvious that $d\Delta/dz$ monotonically decreases for all $\beta \in (0, \pi/2)$. The MNLS soliton perturbed by the intrapulse Raman scattering retain some similarities with the NLS soliton. They both have no shift in initial position and phase. Their energies are not perturbed. They all have frequency redshifts(but in different rates). However, the MNLS soliton has a decrease in amplitude A and an increase in width τ while keeping its energy unperturbed.

4.2 Third-order GVD dispersion

The third-order GVD dispersion[11,12] is described by

$$r(q) = \alpha \partial_t^3 q, \tag{40}$$

where α represents its strength. We have $R_+ = 0$, $R_- = 2\bar{q}_0 r_0$,

$$\frac{d\beta}{dz} = 0, \quad \frac{d\Delta}{dz} = 0, \tag{41}$$

and, using the technique developed in the preceding section,

$$\frac{dt_0}{dz} = \alpha[3s^{-2} - 12(\beta\eta + 4\xi - \beta\xi^2\eta^{-1}) \\ +16(13\eta^2 + \beta\eta\xi - 15\xi^2 + 9\beta\xi^3\eta^{-1})s^2], \tag{42}$$

$$\frac{d\varphi_0}{dz} = \alpha[-2s^{-3} + 12(\beta\eta + 4\xi - 12\beta\xi^2\eta^{-1})s^{-1} \\ +32(-11\eta^2 + \beta\eta\xi + 3\xi^2 - 3\beta\xi^3\eta^{-1})s \\ +64(5\beta\eta^3 - 16\eta^2\xi + 14\beta\eta\xi^2 - 16\xi^3 + 9\beta\xi^4\eta^{-1})s^3]. \tag{43}$$

Within adiabatic approximation, influences of the third-order GVD dispersion on the MNLS soliton is similar to those on the NLS soliton: the amplitude, width, and the main velocity v are unchanged while inducing shifts on t_0 and φ_0. Numerical simulation showed that the third-order GVD dispersion induces radiation emission from the NLS soliton[18] and the problem needs to be solved by a perturbation theory beyond all order[19]. With a similar simulation, one can find similar radiation emission from the MNLS soliton. This means that the problem may also need a perturbation theory beyond all order.

4.3 Narrow band filters with nonlinear gain

For NLS solitons, periodic insertion of narrow band filters was shown to be effective in reducing the frequency shift of solitons. But it was found that filters may induce background instabilities. M. Matsumoto et. al. suggested to utilize nonlinear gains to suppress such instabilities[20]. For further studies, we calculate the adiabatic evolution equations of a MNLS soliton controlled by both of filters and nonlinear gains here. The corresponding perturbation function is

$$r(q) = \Gamma q + \kappa\frac{\partial^2 q}{\partial t^2} + \rho_1|q|^2q + \rho_2|q|^4q, \tag{44}$$

where Γ is an excess gain, $\kappa > 0$ represents the strength of the filter, ρ_1 and ρ_2 are nonlinear gain coefficients. We get $R_+ = 2\bar{q}_0 r_0$, $R_- = 0$, and, with the symbolic technique in the preceding section,

$$\frac{dt_0}{dz} = 0, \qquad \frac{d\varphi_0}{dz} = 0, \tag{45}$$

$$\frac{d\beta}{dz} = -2s^{-2}\kappa\beta - 32s^2\kappa(5\beta\eta^2 - 4\eta\xi + 9\beta\xi^2)$$

$$+2\left[\beta\Gamma + 12\kappa\eta - 16\kappa\beta\xi + 4\rho_1(\eta - 2\beta\xi) + 16\rho_2(2\beta\eta^2 - 3\eta\xi + 6\beta\xi^2)\right], \tag{46}$$

$$\frac{d\Delta}{dz} = \frac{\kappa\Delta(\eta - 2\beta\xi)}{s^2\eta} + \frac{16s^2\kappa\Delta(3\eta^3 - 14\beta\eta^2\xi + 11\eta\xi^2 - 22\beta\xi^3)}{\eta} - \frac{\Delta}{3\eta}\Big[3\eta(\Gamma + 16\kappa\beta\eta) - 3\xi(2\beta\Gamma + 24\kappa\eta) + 144\kappa\beta\xi^2 + 12\rho_1(-3\eta\xi + 6\beta\xi^2 + 2\beta\eta^2) + 32\rho_2(4\eta^3 - 18\beta\eta^2\xi + 15\eta\xi^2 - 30\beta\xi^3)\Big]. \tag{47}$$

5 Summary and discussion

In this letter, we develop a symbolic technique to calculate the adiabatic evolution of perturbed MNLS solitons. Evolution equations under intrapulse Raman scattering, third-order GVD dispersion and narrow band filters with nonlinear gains are calculated with the technique. As $s \to 0$, to keep parameters in the MNLS soliton physically meaningful, ξ and η must be

$$\xi \to \frac{\mu}{2s} + \frac{1}{4s^2}, \quad \eta \to \frac{\nu}{2s}. \tag{48}$$

The MNLS soliton approach the NLS soliton,

$$q_s(t, z) \to -2\nu\,\mathrm{sech}\theta\,\exp(-i\varphi), \tag{49}$$

where

$$\theta = 2\nu(t - \hat{t}), \quad \hat{t} = -2\mu z + t_0, \tag{50}$$

$$\varphi = 4\mu(t - \hat{t}) + \hat{\varphi}, \quad \hat{\varphi} = -4(\mu^2 + \nu^2)z + \varphi_0. \tag{51}$$

In Ref.[16], it was generally shown that evolution equations for perturbed MNLS solitons approach those for perturbed NLS solitons in small s limits. Expanding results in the preceding section near $s = 0$, one can find they do approach their corresponding results for NLS soliton in the literature[11,12,20].

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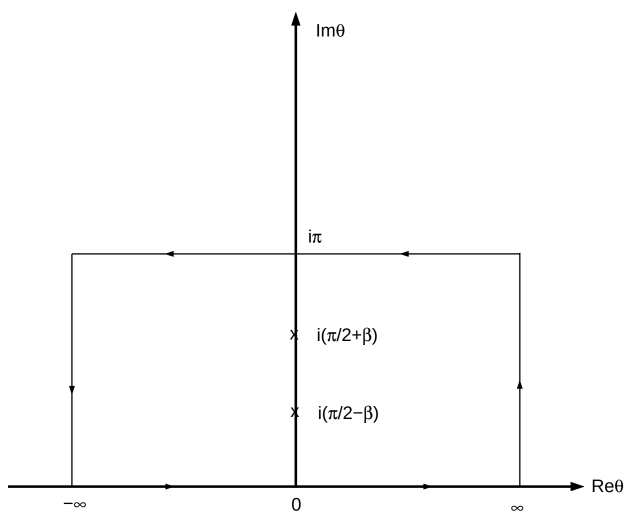

Fig. 1. The path for integrals in section 3. The crosses are the singularities.